Chaos and out-of-time-order correlators in large-N systems

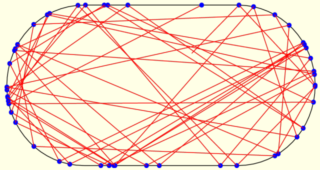
Overview + new results from

Yingfei Gu, AK [1812.00120]

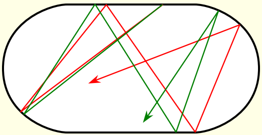
Alexei Kitaev (Caltech)

Introduction: What is chaos?

- A classical example of a chaotic dynamical system is the stadium billiard. Its trajectories are complex and irregular.



- A tiny change in the ball's position or velocity will cause the trajectory to deviate, and this deviation will exponentially grow in time.



- We can see this mathematically or in simulations, but is there a *direct* way to measure chaos? (Problem: no reference data.)

Accessing chaos "experimentally" and theoretically

- **"A flap of a butterfly's wing can change the weather."**

  Is there a *direct* way to test this theory?

  Test 1: Observe current weather. Run the time backward, introduce a butterfly, and run the time forward. Check if the weather is different.

  Test 2: Have two copies of the Earth, with and without the butterfly (but otherwise in the same state).

- Both tests are well-defined in the quantum setting. (Test 2 should be done on the thermofield double.)

- The butterfly effect can be characterized by out-of-time-order correlators like $\langle D(t)C(0)B(t)A(0)\rangle$, where A, B, C, D are some quantum observables.

Naturally ordered (Keldysh) correlators

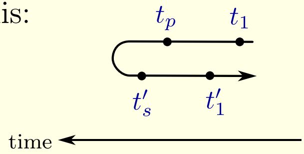
- Consider an abstract quantum experiment setup:
  - The initial state is $\rho = \rho_{\text{system}} \otimes \rho_{\text{probe}}$
  - The system and the probe interact and evolve *forward in time*:

$$H = H_{\text{system}} + H_{\text{probe}} - \sum_j \underset{\substack{\uparrow \\ \text{system}}}{X_j}\, \underset{\substack{\nwarrow \\ \text{probe}}}{Y_j}, \qquad U = \underset{\substack{\uparrow \\ \text{time ordering}}}{\mathbf{T}} \exp\left(-i\int H(t)dt\right)$$

  - A yes-or-no measurement is performed, producing the "yes" outcome with probability $P = \operatorname{Tr}\big(U^\dagger \Pi U \rho\big)$.

- By evaluating various quantities on the probe side, the probability P expands into terms like this:

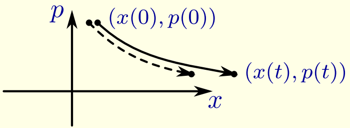
$$\boxed{\begin{gathered}\big\langle X_{j_1}(t_1')^\dagger \cdots X_{j_s}(t_s')^\dagger\, X_{k_p}(t_p) \cdots X_{k_1}(t_1)\big\rangle \\ t_1' < \cdots < t_s', \quad t_p > \cdots > t_1\end{gathered}}$$



Out-of-time correlators (OTOCs)

- Correlators with alternating times do not appear in the previous setting. They require either "time travel" or a TFD.

- Semiclassical behavior in Hamiltonian systems, $H = H(\vec{x}, \vec{p})$
  $\vec{x}(t), \vec{p}(t)$ depend on the initial conditions $\vec{x}(0), \vec{p}(0)$:



$$[p_j(t), p_k(0)] = i\hbar \frac{\partial p_j(t)}{\partial x_k(0)}$$

Lyapunov exponent

  – Chaotic systems: $[p_j(t), p_k(0)] \sim \hbar e^{\lambda t}$,

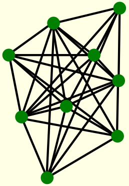
$$\boxed{\left\langle [p_j(t), p_k(0)]^2 \right\rangle \sim \hbar^2 e^{2\lambda t}}$$

small parameter

$\langle p_j(t) p_k(0) p_j(t) p_k(0) \rangle + 3$ other terms

Larkin, Ovchinnikov (1969)

Large N systems with all-to-all interactions



- Random Heisenberg model (N spins)

$$H = -\sum_{j<k} \sum_{\alpha} J_{jk} S_j^{\alpha} S_k^{\alpha}, \qquad \overline{J_{jk}^2} = \frac{J^2}{N}$$

Random parameters

- SYK model (N Majorana modes)

$$H = -\frac{1}{4!} \sum_{j,k,l,m} J_{jklm}\, \chi_j\, \chi_k\, \chi_l\, \chi_m, \qquad \overline{J_{jklm}^2} = 3!\,\frac{J^2}{N^3}$$

  – Sachdev, Ye, 1992 – a similar model with $\mathrm{SU}(M)$ spins and two-body interactions. (The spins are made of fermions.)

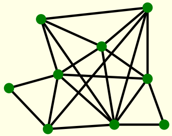
  – This model (Kitaev, 2015)

  – Detailed calculations: Maldacena, Stanford, `arxiv:1604.07818`, Kitaev, Suh, `arXiv:1711.08467`

The SYK model: full definition

N Majorana operators χ_j

$$\dim \mathcal{H} = 2^{N/2}$$



Operator algebra:

$$\chi_j \chi_k + \chi_k \chi_j = \delta_{jk}$$

antisymmetric tensor

$$\boxed{\begin{gathered} H = -\frac{1}{4!} \sum_{j,k,l,m} J_{jklm}\, \chi_j\, \chi_k\, \chi_l\, \chi_m \\ \overline{J_{jklm}} = 0, \qquad \overline{J_{jklm}^2} = 3!\frac{J^2}{N^3} \end{gathered}}$$

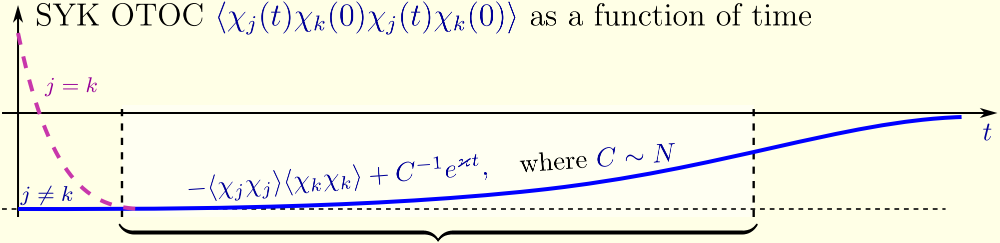
The coupling parameters J_{jklm} are Gaussian random variables (independent for $j < k < l < m$)

– This is the $q = 4$ variant. More generally, one can consider interactions of order $q = 2, 4, 6, \ldots$, though the $q = 2$ case is degenerate.

– For $N \gg 1, \beta$, the model is solved by the *dynamical mean field* method.

OTOCs in non-integrable large N systems

SYK OTOC $\langle \chi_j(t)\chi_k(0)\chi_j(t)\chi_k(0)\rangle$ as a function of time



"Early times": $\varkappa^{-1} \ll t \ll t_{\text{scr}} = \varkappa^{-1}\ln N$

- Generic behavior: $\langle D(t)C(0)B(t)A(0)\rangle - \langle DB\rangle\langle CA\rangle \sim \frac{1}{N}\, e^{\varkappa t}$
  - In most examples, $\varkappa \ll 2\pi T$.
  - In general, $\boxed{\varkappa \leqslant 2\pi T}$ (Shenker, Stanford, and Maldacena, 2015).
  - For the SYK model, $\varkappa = 2\pi T\big(1 - O\big(\frac{T}{J}\big)\big)$.

Qualitative explanation of the exponential growth

- Consider the SYK model at infinite temperature. Express $\chi_j(t)$ as

$$\chi_j(t) = \sum_s c_s \chi_s(0) + \sum_{s_1<s_2<s_3} c_{s_1 s_2 s_3} \chi_{s_1}(0)\chi_{s_2}(0)\chi_{s_3}(0) + \cdots$$

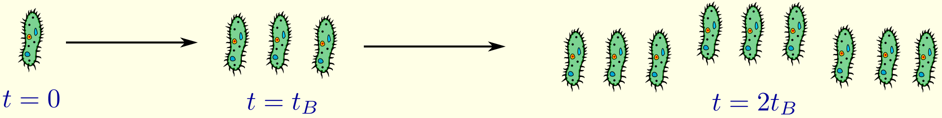
Then

$$\sum_S |c_S|^2 = 1, \qquad \left\langle \{\chi_j(t), \chi_k(0)\}^2 \right\rangle = \sum_{S \ni k} |c_S|^2 = \frac{1}{N} \langle \text{size of } S \rangle$$

support set

- Heisenberg evolution: $\frac{d}{dt}\chi_j = i[H, \chi_j] = \frac{i}{3!} \sum_{k,l,m} J_{jklm} \chi_k \chi_l \chi_m$.

$\langle \text{size of } S \rangle$ grows exponentially until it becomes $\sim N$.



OTOC and the general four-point function

- The OTOC $\langle D(t)B(0)C(t)A(0)\rangle$ is a special case of

$$\langle D(\theta_1)B(\theta_3)C(\theta_2)A(\theta_4)\rangle$$

$\theta_1 \approx \theta_2 \approx it$ $\qquad$ $\theta_3 \approx \theta_4 \approx 0$

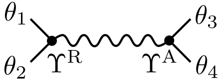
– We use the complex time variable $\quad \theta = it + \tau \quad (0 < \tau < \beta)$

– For convenience, $\quad \beta = 2\pi$

– Thus, the bound on the Lyapunov exponent reads: $\quad 0 \leqslant \varkappa \leqslant 1$

Single-mode anzats for early-time OTOCs

(Kitaev, Suh [arxiv:1711.08467])



$\theta_1 \approx \theta_2 \approx it \qquad \theta_3 \approx \theta_4 \approx 0$

$\theta = \tau + it, \quad 0 < \tau < 2\pi$

- Connected OTOC (like $\langle D(t)C(0)B(t)A(0)\rangle - \langle DC\rangle\langle BA\rangle$):

$$\Big\langle \chi_j(\theta_1)\chi_k(\theta_3)\chi_j(\theta_2)\chi_k(\theta_4)\Big\rangle + \langle\chi_j\chi_j\rangle\langle\chi_k\chi_k\rangle$$
$$\approx C^{-1} e^{i\varkappa(\pi-\theta_1-\theta_2+\theta_3+\theta_4)/2}\,\Upsilon^{\mathrm{R}}(\theta_1-\theta_2)\,\Upsilon^{\mathrm{A}}(\theta_3-\theta_4)$$

- (Anti)-commutaor OTOC (like $\langle[p(t),p(0)]^2\rangle$):

$$\Big\langle \big\{\chi_j(\theta_1),\chi_k(\theta_3)\big\}\big\{\chi_j(\theta_1),\chi_k(\theta_3)\big\}\Big\rangle$$
$$\approx \frac{2\cos(\varkappa\pi/2)}{C}\, e^{-i\varkappa(\theta_1+\theta_2-\theta_3-\theta_4)/2}\,\Upsilon^{\mathrm{R}}(\theta_1-\theta_2)\,\Upsilon^{\mathrm{A}}(\theta_3-\theta_4)$$

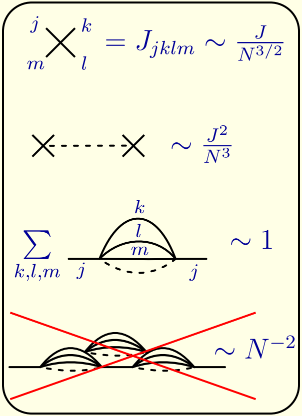
Outline of the technical part

- Dynamical mean-field (illustrated by the SYK model)
- Kinetic equation for the early-time OTOCs
- *Ladder identity* and *branching time*
- Some applications:
  - Near-maximal chaos at $\beta J \gg 1$
  - Maximal chaos in the butterfly wavefront

SYK model: the Green function

Definition of the imaginary-time Green function:

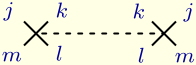
$$\boxed{G(\tau_1, \tau_2) = -\langle \mathbf{T}\, \chi_j(\tau_1)\chi_j(\tau_2)\rangle}$$



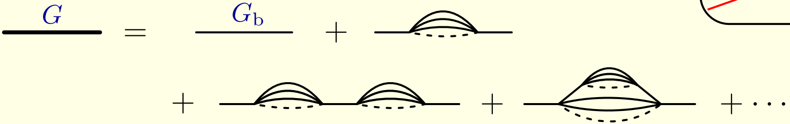
Bare Green function (for $H = 0$):

$$\hat{G}_\mathrm{b} = (-\partial_\tau)^{-1}, \quad G_\mathrm{b}(\tau_1, \tau_2) = -\tfrac{1}{2}\,\mathrm{sgn}(\tau_1 - \tau_2)$$

Disorder-averaged interaction:

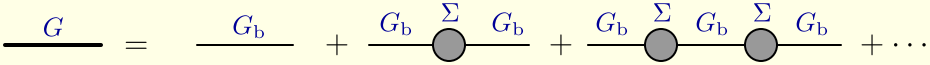


Taylor expansion in the interaction strength βJ:



SYK model: the Schwinger-Dyson equations

- General structure of the Green function:



- Schwinger-Dyson equations:

$(-\partial_\tau - \hat{\Sigma})\hat{G} = 1$ i.e. $(\Sigma G)(\tau_1, \tau_2) = \int d\tau\, \Sigma(\tau_1, \tau) G(\tau, \tau_2)$,

i.e. $\Sigma(\tau_1, \tau_2) = J^2 G(\tau_1, \tau_2)^{q-1}$

The second equation is a variant of the dynamic mean-field.

Solution at long times $(|\tau_1 - \tau_2| \gg J^{-1})$

negligible

- Solving the equations $\quad (-\partial_\tau - \hat{\Sigma})\hat{G} = 1, \qquad \Sigma(\tau_1, \tau_2) = J^2 G(\tau_1, \tau_2)^{q-1}$

- Solution for the zero temperature $\quad (\beta = \infty) \quad$ (Sachdev, Ye 1992)

$$\boxed{G_{\beta=\infty}(\tau_1, \tau_2) \approx -b^{\Delta} \left| J(\tau_1 - \tau_2) \right|^{-2\Delta} \operatorname{sgn}(\tau_1 - \tau_2)} \qquad \boxed{\Delta = 1/q}$$

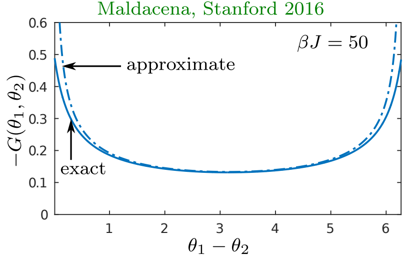
- At finite temperature, let $\quad \theta = \frac{2\pi}{\beta}\,\tau$

If $\beta J \gg 1$, then

$$G(\theta_1, \theta_2) \propto \theta_{12}^{-2\Delta} \operatorname{sgn}\theta_{12}$$

where $\quad \theta_{12} = 2\sin\frac{\theta_1 - \theta_2}{2}$

(Parcollet, Georges 1998)



OTOC and the general four-point function

- The OTOC $\langle D(t)B(0)C(t)A(0)\rangle$ is a special case of

$$\langle D(\theta_1)B(\theta_3)C(\theta_2)A(\theta_4)\rangle$$

$\theta_1 \approx \theta_2 \approx it$ $\qquad$ $\theta_3 \approx \theta_4 \approx 0$

– We use the complex time variable $\quad \theta = it + \tau \quad (0 < \tau < \beta)$

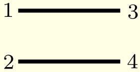
– For convenience, $\quad \beta = 2\pi$

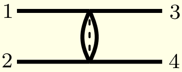
– Thus, the bound on the Lyapunov exponent reads: $\quad 0 \leqslant \varkappa \leqslant 1$

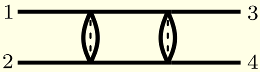
Connected four-point function $\mathcal{F}$

$$\left\langle \mathbf{T}\, \chi_j(\theta_1)\chi_j(\theta_2)\chi_k(\theta_3)\chi_k(\theta_4)\right\rangle = G(\theta_1,\theta_2)G(\theta_3,\theta_4) + \frac{1}{N}\mathcal{F}(\theta_1,\theta_2,\theta_3,\theta_4)$$

- Diagrammatic expansion (up to subleading $1/N$ terms)

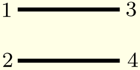
$\mathcal{F}(\theta_1,\theta_2,\theta_3,\theta_4) = -$ 

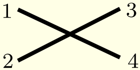
$-$ 

$-$ 

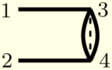
$- \ldots \quad + (3 \leftrightarrow 4)$

- Bethe-Salpeter equation: $\mathcal{F} = \mathcal{F}_0 + K\mathcal{F}$

$\mathcal{F}_0(\theta_1,\theta_2,\theta_3,\theta_4) = -$ 

$+$ 

$\theta_{ab} := \theta_a - \theta_b$

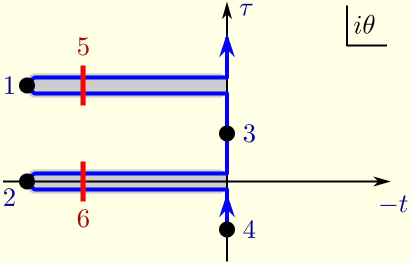
$K(\theta_1,\theta_2,\theta_3,\theta_4) = -$ 

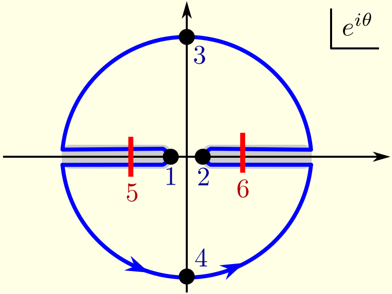
$= -J^2(q-1)G(\theta_{13})G(\theta_{24})G(\theta_{34})^{q-2}$

Connected OTOC

$$\begin{aligned}\mathrm{OTOC}(t_1,t_2,t_3,t_4) &= -N^{-1}\mathcal{F}(\theta_1,\theta_2,\theta_3,\theta_4)\\ &= \left\langle \chi_j(\theta_1)\chi_k(\theta_3)\chi_j(\theta_2)\chi_k(\theta_4)\right\rangle + \langle\chi_j\chi_j\rangle\langle\chi_k\chi_k\rangle\end{aligned}$$

$$\theta_1 = it_1+\pi, \qquad \theta_2 = it_2, \qquad \theta_3 = it_3+\tfrac{\pi}{2}, \qquad \theta_4 = it_4-\tfrac{\pi}{2}$$



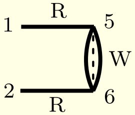


$$\mathcal{F}(\theta_1,\theta_2,\theta_3,\theta_4) \approx \int_{\text{folds}} d\theta_5\, d\theta_6\, K(\theta_1,\theta_2,\theta_5,\theta_6)\,\mathcal{F}(\theta_5,\theta_6,\theta_3,\theta_4)$$

Kinetic equation and retarded kernel

- Let $F(t_1, t_2) = \mathrm{OTOC}(t_1, t_2, t_3, t_4)$. Then

$$\boxed{F(t_1, t_2) = \int_{\mathbb{R}} dt_5\, dt_6\, K^{\mathrm{R}}(t_1, t_2, t_5, t_6)\, F(t_5, t_6)}$$

$K^{\mathrm{R}}(t_1, t_2, t_5, t_6) = -$ 

$= -J^2(q-1) G^{\mathrm{R}}(t_{15}) G^{\mathrm{R}}(t_{26}) G^{\mathrm{W}}(t_{56})^{q-2}$

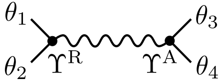
- Eigenfunctions: $\boxed{\widetilde{\Upsilon}_\alpha(t_1, t_2) = e^{-\alpha(t_1+t_2)/2}\,\Upsilon(t_1 - t_2)}$

$$K^{\mathrm{R}} \widetilde{\Upsilon}_\alpha = k_{\mathrm{R}}(\alpha) \widetilde{\Upsilon}_\alpha \quad \Leftrightarrow \quad K^{\mathrm{R}}_\alpha \Upsilon_\alpha = k_{\mathrm{R}}(\alpha) \Upsilon_\alpha \,,$$

where $K^{\mathrm{R}}_\alpha(t, t') = \int K^{\mathrm{R}}\big(s + \frac{t}{2}, s - \frac{t}{2}, \frac{t'}{2}, -\frac{t}{2}\big)\, e^{\alpha s}\, ds$

- Solving for the Lyapunov exponent: $\boxed{k_{\mathrm{R}}(-\varkappa) = 1}$

Connection with the anzats



$\theta_1 \approx \theta_2 \approx it \qquad \theta_3 \approx \theta_4 \approx 0$

$\theta = \tau + it, \quad 0 < \tau < 2\pi$

$$\left\langle \chi_j(\theta_1)\chi_k(\theta_3)\chi_j(\theta_2)\chi_k(\theta_4)\right\rangle + \langle\chi_j\chi_j\rangle\langle\chi_k\chi_k\rangle$$

$$\approx C^{-1} e^{i\varkappa(\pi-\theta_1-\theta_2+\theta_3+\theta_4)/2}\, \Upsilon^{\mathrm{R}}(\theta_1-\theta_2)\, \Upsilon^{\mathrm{A}}(\theta_3-\theta_4)$$

seemingly independent coefficient

solution to the eigenvalue equation

eigenfunction of $K^{\mathrm{R}}_{-\varkappa}$

eigenfunction of $K^{\mathrm{A}}_{-\varkappa}$

$$\Big\langle \big\{\chi_j(\theta_1), \chi_k(\theta_3)\big\}\big\{\chi_j(\theta_1), \chi_k(\theta_3)\big\}\Big\rangle$$

$$\approx \frac{2\cos(\varkappa\pi/2)}{C}\, e^{-i\varkappa(\theta_1+\theta_2-\theta_3-\theta_4)/2}\, \Upsilon^{\mathrm{R}}(\theta_1-\theta_2)\, \Upsilon^{\mathrm{A}}(\theta_3-\theta_4)$$

Example: SYK model for $N \gg \beta J \gg 1$

- The model is maximally chaotic: $\varkappa \approx 1$.
- The eigenfunctions $\widetilde{\Upsilon}^{\mathrm{R}}_{-\varkappa}$ and $\widetilde{\Upsilon}^{\mathrm{A}}_{-\varkappa}$ are generated by the action of

$$L_{-1} = e^{t}(\partial_t + \Delta), \qquad L_1 = e^{-t}(\partial_t - \Delta)$$

on the first variable of the Wightman function

$$G^{\mathrm{W}}(t_1, t_2) = G(it_1 + \pi, it_2)\,.$$

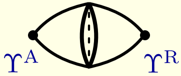
$$\mathrm{OTOC}(t_1, t_2, t_3, t_4) \approx \frac{e^{(t_1+t_2-t_3-t_4)/2}}{C}\Upsilon^{\mathrm{R}}(t_{12})\Upsilon^{\mathrm{A}}(t_{34}), \qquad C = \frac{4\pi\alpha_S N}{\beta J}$$

(C is obtained from the Schwarzian theory)

Main results

- Ladder identity:

$$\boxed{N\,\frac{2\cos\frac{\varkappa\pi}{2}}{C}\,k_{\mathrm{R}}'(-\varkappa)\,(\Upsilon^{\mathrm{A}},\Upsilon^{\mathrm{R}})=1} \qquad (\Upsilon^{\mathrm{A}},\Upsilon^{\mathrm{R}}) =$$

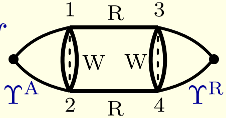


  – Allows for the calculation of C from the retarded kernel;

  – Conversely, in the case of near-maximal chaos, one can calculate $\delta\varkappa = 1-\varkappa$ using $k_{\mathrm{R}}'(-1)$ from the conformal limit and C from the Schwarzian theory.

- *Branching time* $t_B = k_{\mathrm{R}}'(-\varkappa)$ is the average time separation s between adjacent rungs in a ladder diagram contributing to the OTOC:

$$t_B = \frac{1}{(\Upsilon^{\mathrm{A}},\Upsilon^{\mathrm{R}})}\int$$

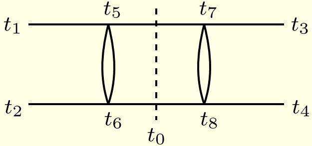


$$dt_{12}\,dt_{34}\,s\,ds, \qquad s = \frac{t_1+t_2}{2} - \frac{t_3+t_4}{2}$$

Derivation sketch

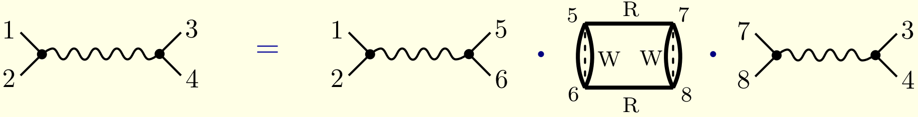
- Idea: cut a long ladder in half; find a consistency condition.

– Cuting the ladder: Fix t_0; find adjacent rungs such that



$$\frac{t_5+t_6}{2} < t_0 < \frac{t_7+t_8}{2}$$

– Consistency condition:



– The factor $2\cos\frac{\varkappa\pi}{2} = e^{i\varkappa\pi/2} + e^{-i\varkappa\pi/2}$ arises because there are two different ways to put θ_5, θ_6 on the double Keldysh contour.

Near-maximal chaos $(\beta J \to \infty, \quad \varkappa \to 1)$

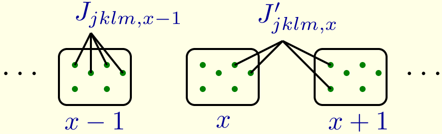
- The prefactor $r = \dfrac{2\cos(\varkappa\pi/2)}{C}$ in the commutator OTOC has a finite limit:

$$r = \Big(k_{\mathrm{R}}'(-1)\left(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{A}}\right)\Big)^{-1} N^{-1}$$

- The correction to the Lyapunov exponent is

$$1 - \varkappa \approx \frac{rC}{\pi} = \frac{2\alpha_S}{\pi k_{\mathrm{R}}'(-1)\left(\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{A}}\right)}\, J^{-1}$$

Application to a 1D model



(Gu, Qi, Stanford 2016)

- $\mathrm{OTOC}_{x,0}(t_1,t_2,t_3,t_4) := \big\langle \chi_{j,x}(\theta_1)\chi_{k,0}(\theta_3)\chi_{j,x}(\theta_2)\chi_{k,0}(\theta_4)\big\rangle + \langle\cdots\rangle\langle\cdots\rangle$

- Fourier transform: $\displaystyle\int \frac{dp}{2\pi} e^{ipx} \underbrace{\mathrm{OTOC}_p(t_1,t_2,t_3,t_4)}_{\propto C(p)^{-1} e^{\varkappa(p)t}}$ $\quad t = \frac{t_1+t_2}{2} - \frac{t_3+t_4}{2}$

  $\varkappa(p) \approx \varkappa(0) - t_B a p^2$ is equal to 1 at some $p_1 = i|p_1|$,

  hence $C(p)^{-1} = \big(N \cdot 2\cos\frac{\varkappa(p)\pi}{2} \cdot t_B \cdot (\Upsilon^{\mathrm{A}}, \Upsilon^{\mathrm{R}})\big)^{-1}$ has a pole.

- Result: The Lyapunov exponent *in the butterfly wavefront* is exactly 1 is J is above threshold.

Summary and further ideas

- The ladder identity is very useful for calculating OTOCs.
- The commutator OTOC is proportional to t_B^{-1} and characterizes dissiparive effects.
  - Such effects admit an interpretation as inelastic scattering in a certain effective model
  - Analogous to grvitational scattering of massive particles near a black hole horizon, where the dissipative effects are due string production (Shenker, Stanford 2014)
  - Challenge: construct a model with $t_B \gg 1$. Such a model might have some virtues of string theory